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Convex-like inequality, homogeneity, subadditivity, and a characterization of L^p -norm

by JANUSZ MATKOWSKI and MAREK PYCIA (Bielsko-Biała)

Abstract. Let a and b be fixed real numbers such that $0 < \min\{a, b\} < 1 < a + b$. We prove that every function $f : (0, \infty) \rightarrow \mathbb{R}$ satisfying $f(as + bt) \leq af(s) + bf(t)$, $s, t > 0$, and such that $\limsup_{t \rightarrow 0+} f(t) \leq 0$ must be of the form $f(t) = f(1)t$, $t > 0$. This improves an earlier result in [5] where, in particular, f is assumed to be nonnegative. Some generalizations for functions defined on cones in linear spaces are given. We apply these results to give a new characterization of the L^p -norm.

Introduction. We deal with the functional inequality

$$f(as + bt) \leq af(s) + bf(t),$$

where $a, b \in \mathbb{R}$ are fixed real numbers such that

$$(1) \quad 0 < \min\{a, b\} < 1 < a + b$$

and f is a real function defined on $\mathbb{R}_+ := [0, \infty)$ or $(0, \infty)$. Our Theorem 2 says that if $f(0) = 0$, f is bounded above in a neighbourhood of 0, and satisfies this inequality for all $s, t \geq 0$, then f must be a linear function. This improves a result of [6] where f is assumed to be nonnegative. Theorem 1, the main result of the first section, reads as follows: *If $f : (0, \infty) \rightarrow \mathbb{R}$ satisfies the above inequality for all $s, t > 0$, and $\limsup_{t \rightarrow 0+} f(t) \leq 0$, then $f(t) = f(1)t$, $t > 0$.*

In Section 2, using Theorems 1 and 2, we obtain their counterparts for functions defined on convex cones of a linear space. Namely, under some weak regularity conditions an analogue of the above inequality characterizes the Banach functionals.

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Applying these results we give a new characterization of the L^p -norm (cf. Theorem 3).

1. Functions satisfying a convex-like inequality on $(0, \infty)$ and \mathbb{R}_+ . The main theorem of this section is a refinement of a relevant result of [6] and reads as follows:

THEOREM 1. *Let $a, b \in \mathbb{R}$ be fixed and such that condition (1) holds. If $f : (0, \infty) \rightarrow \mathbb{R}$ satisfies*

$$(2) \quad f(as + bt) \leq af(s) + bf(t), \quad s, t > 0,$$

and

$$(3) \quad \limsup_{t \rightarrow 0+} f(t) \leq 0,$$

then $f(t) = f(1)t$, $t > 0$.

Proof. There is no loss of generality in assuming that $a = \min\{a, b\} < 1$. Moreover, by (2),

$$f(as + b(a + b)^n t) \leq af(s) + b(a + b)^n f(t), \quad s, t > 0, \quad n \in \mathbb{N}.$$

Consequently, we may also assume $b > 1$. Now we prove the following

CLAIM. *Under the conditions of Theorem 1 and $a < 1 < b$ there exists an $M > 0$ such that*

$$(4) \quad ka^n b^m f(t) + M\delta \geq f(ka^n b^m t + \delta),$$

for all $t, \delta > 0$; $n, m \in \mathbb{N}$, $n + m > 0$; $k = 0, \dots, \binom{n+m}{m}$.

To show it, take $c > \max\{a + b, a^{-1}\}$. By (3) there exists a $t_0 > 0$ such that f is bounded above on the interval $I := (t_0, ct_0)$. Thus, for some $M > 0$,

$$(5) \quad f(t) \leq Mt, \quad t \in I.$$

From (2), $f((a + b)^n t) \leq (a + b)^n f(t)$ for all $n \in \mathbb{N}$ and $t > 0$. Hence

$$f(t) \leq Mt, \quad t \in \bigcup_{n=0}^{\infty} (a + b)^n I.$$

(For $I \subset \mathbb{R}$ and $\lambda \in \mathbb{R}$ we denote by λI the set $\{\lambda x : x \in I\}$.) Since $c > a + b$, the intervals $(a + b)^n I$ and $(a + b)^{n+1} I$ have a nonempty intersection, and, consequently, $\bigcup_{n=0}^{\infty} (a + b)^n I = (t_0, \infty)$. This proves that $f(t) \leq Mt$ for all $t \in (t_0, \infty)$.

Assume that for some $n \in \mathbb{N}$,

$$f(t) \leq Mt, \quad t \in a^n I,$$

and take $s \in a^{n+1}I$. There exists an increasing sequence (t_k) such that $t_k \in a^n I$ ($k \in \mathbb{N}$), and $at_k \rightarrow s$. From (2) we have

$$\begin{aligned} f(s) &= f(at_k + bb^{-1}(s - at_k)) \leq af(t_k) + bf(b^{-1}(s - at_k)) \\ &\leq Mat_k + bf(b^{-1}(s - at_k)). \end{aligned}$$

According to (3),

$$f(s) \leq Ma(\lim_{k \rightarrow \infty} t_k) = Ms, \quad s \in a^{n+1}I.$$

Hence, by induction,

$$f(s) \leq Ms, \quad s \in \bigcup_{n=0}^{\infty} a^n I.$$

Since the inequality $c > a^{-1}$ implies that $\bigcup_{n=0}^{\infty} a^n I = (0, ct_0)$, it follows that $f(t) \leq Mt$, $t \in (0, ct_0)$. Thus we have proved

$$(6) \quad f(t) \leq Mt, \quad t > 0.$$

We now show (4) by induction on $N := n + m$. For $N = 1$, (4) follows immediately from (2) and (6), for $k = 0$ it reduces to (6). Take $N > 1$, $k > 0$, choose k_1, k_2 such that

$$k_1 + k_2 = k, \quad k_1 \leq \binom{n+m-1}{m}, \quad k_2 \leq \binom{n+m-1}{m-1},$$

and suppose that

$$\begin{aligned} k_1 a^{n-1} b^m f(t) + (2a)^{-1} \delta &\geq f(k_1 a^{n-1} b^m t + (2a)^{-1} \delta), \\ k_2 a^n b^{m-1} f(t) + (2a)^{-1} \delta &\geq f(k_2 a^n b^{m-1} t + (2a)^{-1} \delta). \end{aligned}$$

Hence, in view of (2), we get

$$\begin{aligned} ka^n b^m f(s) + M\delta &= a(k_1 a^{n-1} b^m f(s) + M(2a)^{-1} \delta) + b(k_2 a^n b^{m-1} f(s) + M(2a)^{-1} \delta) \\ &\geq af(k_1 a^{n-1} b^m s + (2a)^{-1} \delta) + bf(k_2 a^n b^{m-1} s + (2a)^{-1} \delta) \\ &\geq f(ak_1 a^{n-1} b^m s + 2^{-1} \delta + bk_2 a^n b^{m-1} s + 2^{-1} \delta) = f(ka^n b^m s + \delta), \end{aligned}$$

and the induction completes the proof of our claim.

Now note that the set

$$\mathbb{D} := \left\{ ka^n b^m : m, n \in \mathbb{N}, m + n > 0, k = 0, \dots, \binom{n+m}{m} \right\}$$

is dense in $(0, \infty)$. Indeed, if $\log b / \log a$ is irrational, then, in view of Kronecker's Theorem, its subset $\{a^{n+1} b^m : m, n \in \mathbb{N}\}$ is dense in $(0, \infty)$. In the other case there exist $n, m \in \mathbb{N}$ such that $\log b / \log a = -n/m$, which means

that $a^n b^m = 1$. Since for every $k, j \in \mathbb{N}$,

$$ka^j b = ka^{kn+j} b^{km+1} \in \mathbb{D},$$

the set \mathbb{D} contains a dense subset $\{ka^j b : k, j \in \mathbb{N}\}$.

By the definition of \mathbb{D} we can write (4) in the following equivalent form:

$$(7) \quad \lambda f(t) + M\delta \geq f(\lambda t + \delta), \quad \lambda \in \mathbb{D}, \quad t, \delta > 0.$$

Now, fix $s, t > 0$ and take a sequence (λ_n) such that $\lambda_n \in \mathbb{D}$, $\lambda_n < s$ ($n \in \mathbb{N}$), $\lim_{n \rightarrow \infty} \lambda_n = s$. From (7) we have

$$\lambda_n f(t) + M(s - \lambda_n)t \geq f(\lambda_n t + (s - \lambda_n)t) = f(st), \quad n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ we obtain $sf(t) \leq f(st)$, which obviously implies that $sf(t) = f(st)$. Hence $f(s) = f(1)s$, $s > 0$, which completes the proof.

Remark 1. It is shown in [6] that *every nonnegative function f satisfying (2) with a, b such that (1) holds must be linear*. Obviously, this result is a consequence of Theorem 1.

EXAMPLE 1. Take $a, b > 0$ such that $a + b > 1$, and $c > 0$. Then every function $f : (0, \infty) \rightarrow \mathbb{R}$ such that $c \leq f(t) \leq c(a + b)$, $t > 0$, satisfies (2). This shows that the condition (3) in Theorem 1 is essential.

Note that (3) can be considerably weakened if (2) is assumed to hold for all nonnegative s and t . Namely, we have the following

THEOREM 2. *Let $a, b \in \mathbb{R}$ satisfy (1). If $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies*

$$f(as + bt) \leq af(s) + bf(t), \quad s, t \geq 0,$$

and

- (i) $f(0) = 0$;
- (ii) f is bounded above in a right vicinity of 0,

then $f(t) = f(1)t$, $t > 0$.

This result is an immediate consequence of Theorem 1 and the following

LEMMA 1. *Let $a, b \in \mathbb{R}$ satisfy (1). Suppose that $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies*

$$f(as + bt) \leq af(s) + bf(t), \quad s, t \geq 0.$$

Then

- (i) $f(0) \geq 0$.
- (ii) *If, moreover, $f(0) = 0$ and f is bounded above in a right vicinity of 0, then condition (3) holds.*

Proof. (i) is obvious. To prove (ii) suppose that, say, $a = \min\{a, b\}$ and observe that, by the boundedness above of f to the right of 0, we have

$$c := \limsup_{t \rightarrow 0+} f(t) < \infty.$$

Setting in the assumed inequality $s = 0$ and making use of the condition $f(0) = 0$, we get $f(at) \leq af(t)$ for all $t \geq 0$. It follows that $c \leq ac$. Since $a < 1$ we hence get $c \leq 0$, which was to be shown.

EXAMPLE 2. The function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by $f(t) = t^{-1}$, $t > 0$, and $f(0) = 0$ satisfies (2) for all $a, b \in \mathbb{R}$ such that condition (1) holds. This shows that, in Theorem 2, the assumption of f being bounded above in a (right) neighbourhood of 0 is indispensable.

EXAMPLE 3. Let $a, b > 0$ be rational. Then every discontinuous additive function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies (2). It is well known that the graph of f is a dense subset of the plane (cf. for instance Aczél–Dhombres [1], p. 14). This also shows that the regularity assumptions in Theorems 1 and 2 are necessary.

2. Some generalizations for functions defined on cones. In this section, using Theorems 1 and 2, we prove their more general counterparts.

Let \mathbf{X} be a real linear space. A set $\mathbf{C} \subset \mathbf{X}$ is said to be a *convex cone* in \mathbf{X} iff $\mathbf{C} + \mathbf{C} \subset \mathbf{C}$ and $t\mathbf{C} \subset \mathbf{C}$ for all $t > 0$.

A functional $\mathbf{p} : \mathbf{C} \rightarrow \mathbb{R}$ is called *subadditive* iff

$$\mathbf{p}(\mathbf{x} + \mathbf{y}) \leq \mathbf{p}(\mathbf{x}) + \mathbf{p}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbf{C},$$

and *positively homogeneous* iff

$$\mathbf{p}(t\mathbf{x}) = t\mathbf{p}(\mathbf{x}), \quad t > 0, \mathbf{x} \in \mathbf{C}.$$

In the sequel the functionals satisfying both these conditions (the so-called *Banach functionals*) will appear frequently.

Denote by \mathbf{o} the zero vector of \mathbf{X} . If \mathbf{C} is a convex cone in \mathbf{X} and $\mathbf{o} \in \mathbf{C}$, then $t\mathbf{C} \subset \mathbf{C}$ for all $t \geq 0$.

COROLLARY 1. Let \mathbf{X} be a real linear space and $\mathbf{C} \subset \mathbf{X}$ a convex cone such that $\mathbf{o} \in \mathbf{C}$. Suppose that $a, b \in \mathbb{R}$ are fixed and $0 < \min\{a, b\} < 1 < a + b$. Then a function $\mathbf{p} : \mathbf{C} \rightarrow \mathbb{R}$ is subadditive and positively homogeneous if and only if

$$(i) \quad \mathbf{p}(\mathbf{o}) = 0;$$

(ii) for every $\mathbf{x} \in \mathbf{C}$, the function $(0, \infty) \ni t \rightarrow \mathbf{p}(t\mathbf{x})$ is bounded above in a right vicinity of 0; and

$$(8) \quad \mathbf{p}(a\mathbf{x} + b\mathbf{y}) \leq a\mathbf{p}(\mathbf{x}) + b\mathbf{p}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbf{C}.$$

PROOF. First suppose that \mathbf{p} satisfies (i), (ii), and (8). Then for every fixed $\mathbf{x} \in \mathbf{C}$ the function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by $f(t) := \mathbf{p}(t\mathbf{x})$, $t \geq 0$, satisfies all the assumptions of Theorem 2. Consequently, $\mathbf{p}(t\mathbf{x}) = f(t) = f(1)t = t\mathbf{p}(\mathbf{x})$ for all $t \geq 0$, which means that \mathbf{p} is positively homogeneous.

Now the subadditivity of \mathbf{p} is a consequence of (8). Since the converse is obvious, the proof is complete.

In a similar way, applying Theorem 1, we get

COROLLARY 2. *Let \mathbf{X} be a real linear space and $\mathbf{C} \subset \mathbf{X}$ a convex cone. Suppose that $a, b \in \mathbb{R}$ are fixed and $0 < \min\{a, b\} < 1 < a + b$. Then a function $\mathbf{p} : \mathbf{C} \rightarrow \mathbb{R}$ is subadditive and positively homogeneous if and only if it satisfies (8) and*

$$\limsup_{t \rightarrow 0+} \mathbf{p}(t\mathbf{x}) \leq 0, \quad \mathbf{x} \in \mathbf{C}.$$

Let \mathbf{X} be a real linear space, $\mathbf{C} \subset \mathbf{X}$ a convex cone in \mathbf{X} and $\phi : \mathbf{C} \rightarrow \mathbb{R}$. We say that ϕ is a *linear functional* on \mathbf{C} iff $\phi(\mathbf{x} + \mathbf{y}) = \phi(\mathbf{x}) + \phi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{C}$, and $\phi(t\mathbf{x}) = t\phi(\mathbf{x})$ for all $t > 0$, $\mathbf{x} \in \mathbf{C}$. Note that if $\phi \not\equiv 0$, then $\phi^{-1}(\{1\}) = \{\mathbf{x} \in \mathbf{C} : \phi(\mathbf{x}) = 1\}$ is a nonempty convex subset of \mathbf{C} , and put $\text{supp}(\phi) := \{\mathbf{x} \in \mathbf{C} : \phi(\mathbf{x}) \neq 0\}$.

The term “linear functional” is legitimate in view of the following

Remark 2. Let $\phi : \mathbf{C} \rightarrow \mathbb{R}$ be additive and positively homogeneous on a cone $\mathbf{C} \subset \mathbf{X}$ such that $\mathbf{C} \cap (-\mathbf{C}) = \{\mathbf{o}\}$. Denote by \mathbf{Y} the linear span of \mathbf{C} . It is easy to check that there exists a unique linear functional $\Phi : \mathbf{Y} \rightarrow \mathbb{R}$ such that $\Phi|_{\mathbf{C}} = \phi$.

PROPOSITION. *Let \mathbf{X} be a real linear space, $\mathbf{C} \subset \mathbf{X}$ a cone in \mathbf{X} such that $\mathbf{C} \cap (-\mathbf{C}) = \{\mathbf{o}\}$, and $\phi : \mathbf{C} \rightarrow \mathbb{R}$ a linear functional on \mathbf{C} such that $\phi \geq 0$ on \mathbf{C} . Suppose that $a, b \in \mathbb{R}$ are fixed and $0 < \min\{a, b\} < 1 < a + b$. If $H : \text{supp}(\phi) \rightarrow \mathbb{R}$ satisfies*

$$H(a\mathbf{x} + b\mathbf{y}) \leq aH(\mathbf{x}) + bH(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \text{supp}(\phi),$$

and

$$\limsup_{t \rightarrow 0+} H(t\mathbf{x}) \leq 0, \quad \mathbf{x} \in \text{supp}(\phi),$$

then H is positively homogeneous and subadditive.

Moreover, the function $h : \phi^{-1}(1) \rightarrow \mathbb{R}$ defined by

$$h(\mathbf{x}) := H(\mathbf{x}), \quad \mathbf{x} \in \phi^{-1}(1),$$

is convex,

$$H(\mathbf{x}) = \phi(\mathbf{x})h(\mathbf{x}/\phi(\mathbf{x})), \quad \mathbf{x} \in \text{supp}(\phi),$$

and

$$(9) \quad \phi(\mathbf{x} + \mathbf{y})h\left(\frac{\mathbf{x} + \mathbf{y}}{\phi(\mathbf{x} + \mathbf{y})}\right) \leq \phi(\mathbf{x})h\left(\frac{\mathbf{x}}{\phi(\mathbf{x})}\right) + \phi(\mathbf{y})h\left(\frac{\mathbf{y}}{\phi(\mathbf{y})}\right), \quad \mathbf{x}, \mathbf{y} \in \text{supp}(\phi).$$

P r o o f. It is easy to check that $\text{supp}(\phi)$ is a convex cone in \mathbf{X} . Therefore the first conclusion is a consequence of Corollary 2.

To prove the remaining assertion note that $\mathbf{z} \in \phi^{-1}(1)$ if and only if there is an $\mathbf{x} \in \text{supp}(\phi)$ such that $\mathbf{z} = \mathbf{x}/\phi(\mathbf{x})$. Take any $\mathbf{x} \in \text{supp}(\phi)$. By the positive homogeneity of H and the definition of h we have

$$H(\mathbf{x}) = \phi(\mathbf{x})H(\mathbf{x}/\phi(\mathbf{x})) = \phi(\mathbf{x})h(\mathbf{x}/\phi(\mathbf{x})).$$

Hence, the subadditivity of H gives (9). This inequality implies the convexity of h , and the proof is complete.

R e m a r k 3. Taking in the Proposition $\mathbf{X} = \mathbb{R}^k$, $\mathbf{C} = \mathbb{R}_+^k$, $k \in \mathbb{N}$, and the functional $\phi : \mathbf{C} \rightarrow \mathbb{R}_+$, $\phi(\mathbf{x}) = \phi(x_1, \dots, x_k) = x_i$, the projection on the x_i -axis, $i \in \{1, \dots, k\}$, we get the result proved in [5] (cf. also [6]). Moreover, it is shown in [5] that inequality (9) with ϕ being the projection characterizes the convex functions h defined on $(0, \infty)^{k-1}$ and generalizes Minkowski's and Hölder's inequalities. Thus inequality (9) may also be interpreted as a generalization of these two fundamental inequalities.

3. An application to a characterization of the L^p -norm. For a measure space (Ω, Σ, μ) denote by $\mathbf{S} = \mathbf{S}(\Omega, \Sigma, \mu)$ the linear space of all μ -integrable step functions $\mathbf{x} : \Omega \rightarrow \mathbb{R}$ and by $\mathbf{S}_+ = \mathbf{S}_+(\Omega, \Sigma, \mu)$ the set of all nonnegative $\mathbf{x} \in \mathbf{S}$. If $\varphi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are one-to-one, onto and $\varphi(0) = 0$ then the functional $\mathbf{P}_{\varphi, \psi} : \mathbf{S} \rightarrow \mathbb{R}$ given by the formula

$$\mathbf{P}_{\varphi, \psi}(\mathbf{x}) := \psi\left(\int_{\Omega} \varphi \circ |\mathbf{x}| d\mu\right), \quad \mathbf{x} \in \mathbf{S},$$

is well defined. The goal of this section is to prove the following

T H E O R E M 3. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite and positive measure. Suppose that $a, b \in \mathbb{R}$ are fixed numbers such that*

$$0 < \min\{a, b\} < 1 < a + b,$$

and $\varphi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are one-to-one, onto, continuous at 0 and $\varphi(0) = \psi(0) = 0$. If

$$\mathbf{P}_{\varphi, \psi}(a\mathbf{x} + b\mathbf{y}) \leq a\mathbf{P}_{\varphi, \psi}(\mathbf{x}) + b\mathbf{P}_{\varphi, \psi}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbf{S}_+,$$

then $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^{1/p}$ ($t \geq 0$) for some $p \geq 1$.

P r o o f. Take any $\mathbf{x} \in \mathbf{S}_+$. Then there exist n pairwise disjoint sets $A_1, \dots, A_n \in \Sigma$ of finite measure, and $x_1, \dots, x_n \in \mathbb{R}_+$ such that $\mathbf{x} = \sum_{k=1}^n x_k \chi_{A_k}$. (χ_A stands for the characteristic function of the set A .) From the definition of $\mathbf{P}_{\varphi, \psi}$ we have

$$\mathbf{P}_{\varphi, \psi}(t\mathbf{x}) = \psi\left(\int_{\Omega} \varphi \circ |t\mathbf{x}| d\mu\right) = \psi\left(\sum_{k=1}^n \varphi(tx_k)\mu(A_k)\right), \quad t > 0.$$

The continuity of φ and ψ at zero and $\varphi(0) = \psi(0) = 0$ imply that $\lim_{t \rightarrow 0+} \mathbf{P}_{\varphi, \psi}(t\mathbf{x}) = 0$. By Corollary 2 the functional $\mathbf{P}_{\varphi, \psi}$ is positively homogeneous, i.e.

$$(10) \quad \mathbf{P}_{\varphi, \psi}(t\mathbf{x}) = t\mathbf{P}_{\varphi, \psi}(\mathbf{x}), \quad \mathbf{x} \in S_+, \quad t > 0,$$

and subadditive:

$$(11) \quad \mathbf{P}_{\varphi, \psi}(\mathbf{x} + \mathbf{y}) \leq \mathbf{P}_{\varphi, \psi}(\mathbf{x}) + \mathbf{P}_{\varphi, \psi}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in S_+.$$

By our assumption on the measure space, there are two disjoint sets $A, B \in \Sigma$ of finite positive measure. Put $\alpha := \mu(A)$ and $\beta := \mu(B)$. Taking $\mathbf{x} := x_1\chi_A + x_2\chi_B$ with $x_1, x_2 \geq 0$ in (10), we get

$$\psi(\alpha\varphi(tx_1) + \beta\varphi(tx_2)) = t\psi(\alpha\varphi(x_1) + \beta\varphi(x_2)).$$

Since ψ and φ are bijective we can write this equation in the following equivalent form:

$$(12) \quad \begin{aligned} \alpha\varphi(t\varphi^{-1}(x_1)) + \beta\varphi(t\varphi^{-1}(x_2)) \\ = \psi^{-1}(t\psi(\alpha x_1 + \beta x_2)), \quad t > 0, \quad x_1, x_2 \geq 0. \end{aligned}$$

Substituting here first $x_2 = 0$, and next $x_1 = 0$ we get

$$(13) \quad \alpha\varphi(t\varphi^{-1}(x_1)) = \psi^{-1}(t\psi(\alpha x_1)), \quad t > 0, \quad x_1 \geq 0,$$

$$(14) \quad \beta\varphi(t\varphi^{-1}(x_2)) = \psi^{-1}(t\psi(\beta x_2)), \quad t > 0, \quad x_2 \geq 0.$$

The relations (13) and (14) allow us to write (12) in the form

$$\psi^{-1}(t\psi(\alpha x_1)) + \psi^{-1}(t\psi(\beta x_2)) = \psi^{-1}(t\psi(\alpha x_1 + \beta x_2)), \quad t > 0, \quad x_1, x_2 \geq 0,$$

or, equivalently,

$$\psi^{-1}(t\psi(x_1)) + \psi^{-1}(t\psi(x_2)) = \psi^{-1}(t\psi(x_1 + x_2)), \quad t > 0, \quad x_1, x_2 \geq 0.$$

Thus, for every $t > 0$, the function $\psi^{-1} \circ (t\psi)$ is additive. Since it is non-negative, it follows that for every $t > 0$ there is an $m(t) > 0$ such that

$$(15) \quad \psi^{-1}(t\psi(u)) = m(t)u, \quad u > 0.$$

Writing an analogous equation for every $s > 0$ we have

$$\psi^{-1}(s\psi(u)) = m(s)u, \quad u > 0.$$

Composing separately the functions on the left- and the right-hand sides of these equations we obtain

$$\psi^{-1}(st\psi(u)) = m(s)m(t)u, \quad u > 0.$$

Replacing t by st in (15) we get

$$\psi^{-1}(st\psi(u)) = m(st)u, \quad u > 0.$$

The last two equations imply that $m(st) = m(s)m(t)$, $s, t > 0$, i.e. $m : (0, \infty) \rightarrow (0, \infty)$ is a solution of the multiplicative Cauchy equation. Putting

$u=1$ in (15) we get $m(t)=\psi^{-1}(t\psi(1))$, $t > 0$. It follows that m is a bijection of $(0, \infty)$, and, of course, the inverse function to m ,

$$m^{-1}(t) = \psi(t)/\psi(1), \quad t > 0,$$

is multiplicative. The continuity of ψ at 0 implies that there exists a $p \in \mathbb{R}$, $p \neq 0$, such that $m^{-1}(t) = t^{1/p}$ for all $t > 0$. Hence

$$\psi(t) = \psi(1)t^{1/p}, \quad t > 0.$$

Inserting this into (13) we have $\alpha\varphi(t\varphi^{-1}(x_1)) = \alpha x_1 t^p$ for all $t > 0$ and $x_1 \geq 0$. Taking $x_1 := \varphi^{-1}(1)$ we obtain

$$\varphi(t) = \varphi(1)t^p, \quad t > 0.$$

Now, for the above power functions φ and ψ , (11) reduces to the classical Minkowski inequality. It follows that $p \geq 1$. This completes the proof.

Remark 4. To prove that (13) and (14) imply that φ and ψ are the inverse power functions we could apply some results proved in [4].

A similar result holds if $\mathbf{P}_{\varphi,\psi}$ satisfies the opposite inequality to that of Theorem 3. One should emphasize that, in this case, the regularity assumptions on functions φ and ψ are superfluous. Namely, we have

THEOREM 4. *Let (Ω, Σ, μ) be a measure space with at least two disjoint sets of finite positive measure. Suppose that $a, b \in \mathbb{R}$ are fixed with $0 < \min\{a, b\} < 1 < a + b$, and $\varphi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are one-to-one, onto, and $\varphi(0) = 0$. If*

$$(16) \quad \mathbf{P}_{\varphi,\psi}(a\mathbf{x} + b\mathbf{y}) \geq a\mathbf{P}_{\varphi,\psi}(\mathbf{x}) + b\mathbf{P}_{\varphi,\psi}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbf{S}_+,$$

then $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^{1/p}$ ($t \geq 0$) for some p , $0 < p \leq 1$.

Proof. Since $-\mathbf{P}_{\varphi,\psi}$ satisfies the opposite inequality to (16) and $(-\mathbf{P}_{\varphi,\psi})(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbf{S}_+$, Corollary 2 implies that $\mathbf{P}_{\varphi,\psi}$ is positively homogeneous, and superadditive on \mathbf{S}_+ , i.e.

$$(17) \quad \mathbf{P}_{\varphi,\psi}(\mathbf{x} + \mathbf{y}) \geq \mathbf{P}_{\varphi,\psi}(\mathbf{x}) + \mathbf{P}_{\varphi,\psi}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbf{S}_+.$$

Arguing in the same way as in the proof of Theorem 3 we show that the function $m : (0, \infty) \rightarrow (0, \infty)$, $m(t) = \psi^{-1}[t\psi(1)]$, $t > 0$, is multiplicative on $(0, \infty)$.

As in the proof of Theorem 3, take disjoint sets $A, B \in \Sigma$ of finite positive measure, and put $\alpha := \mu(A)$ and $\beta := \mu(B)$. Substituting, in (17), $\mathbf{x}, \mathbf{y} \in \mathbf{S}_+$ such that

$$\mathbf{x} := x_1\chi_A + x_2\chi_B, \quad \mathbf{y} := y_1\chi_A + y_2\chi_B, \quad x_1, x_2, y_1, y_2 \geq 0,$$

we get

$$\psi(\alpha\varphi(x_1 + y_1) + \beta\varphi(x_2 + y_2)) \geq \psi(\alpha\varphi(x_1) + \beta\varphi(x_2)) + \psi(\alpha\varphi(y_1) + \beta\varphi(y_2))$$

for all $x_1, x_2, y_1, y_2 \geq 0$. Take arbitrary $s, t \geq 0$. Putting

$$x_1 = \varphi(s/\alpha)^{-1}, \quad x_2 = y_1 = 0, \quad y_2 = \varphi(t/\beta)^{-1},$$

and making use of the assumption that $\varphi(0) = 0$, we get

$$\psi(s+t) \geq \psi(s) + \psi(t), \quad s, t \geq 0.$$

Hence ψ is increasing, and, consequently, a homeomorphism of \mathbb{R}_+ . It follows that the multiplicative function m is a homeomorphism of $(0, \infty)$.

Now, by an argument as in the proof of Theorem 3, we show that there exists a $p \in \mathbb{R}$, $p \neq 0$, such that $\psi(t) = \psi(1)t^{1/p}$ and $\varphi(t) = \varphi(1)t^p$, $t > 0$. Substituting these functions into (16) we obtain the “companion” of the Minkowski inequality which is known to hold only for $p \in (0, 1]$. This concludes the proof.

Remark 5. Theorems 3 and 4 can be interpreted to be converses of the Minkowski inequalities (cf. [7] and [8] where converses of Minkowski’s inequality other than Theorem 3 are given).

References

- [1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia Math. Appl. 31, Cambridge University Press, Cambridge, Sydney, 1989.
- [2] E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, Amer. Math. Soc., Providence, R.I., 1957.
- [3] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy’s equation and Jensen’s inequality*, Prace Nauk. Uniw. Śl. 489, Polish Scientific Publishers, 1985.
- [4] J. Matkowski, *On a characterization of L^p -norm*, Ann. Polon. Math. 50 (1989), 137–144.
- [5] —, *A functional inequality characterizing convex functions, conjugacy and a generalization of Hölder’s and Minkowski’s inequalities*, Aequationes Math. 40 (1990), 168–180.
- [6] —, *Functional inequality characterizing nonnegative concave functions in $(0, \infty)$* , ibid. 43 (1992), 219–224.
- [7] —, *The converse of the Minkowski’s inequality theorem and its generalization*, Proc. Amer. Math. Soc. 109 (1990), 663–675.
- [8] —, *L^p -like paranorms*, in: Selected Topics in Functional Equations and Iteration Theory, Proc. Austrian–Polish Seminar, Graz, 1991, Grazer Math. Ber. 316 (1992), 103–135.

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